Quantization condition of resonances at energy-level crossing (Microlocal Analysis and Singular Perturbation Theory)

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Quantization condition of resonances at energy-level crossing

By

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Abstract

We study the asymptotic distribution of semiclassical resonances near an energy-level crossing of the one-dimensional Schrödinger operator with a 2×2 matrix-valued potential. Assuming that this level is in a simple well of one of the eigen-potentials, we deduce a Bohr-Sommerfeld type quantization condition, and show in particular that the width of the resonances is of order $h^{5/3}$ in case of an elliptic interaction while it is of order $h^{7/3}$ in case of a vector field interaction.

§ 1. Introduction

This is a short survey of the papers [FMW1] and [FMW2].

They are concerned with 2×2 semiclassical Schrödinger operator in dimension one, whose matrix-valued potential has eigenvalues crossing transversally at a point, and the aim is to compute the quantization condition of resonances near this crossing level.

This problem comes from the study of diatomic molecular predissociation resonances in the Born-Oppenheimer approximation, at energies close to that of the crossing of the electronic levels. The imaginary part (width) of the resonances corresponds to the inverse of the life-time of the molecule.

The small parameter h stands for the square-root of the inverse of the mass of the nuclei. The Born-Oppenheimer approximation permits to reduce the study to that of a

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semiclassical system of Schrödinger-type operators (see, e.g., [KMSW, MaMe, MaSo]), and the size of the system depends on the number of electronic levels that are involved.

In the system case with multiple electronic levels, only few results are available. One may quote [Ba, Na, FLN, GrMa1], where very particular potentials are considered, and [Kl, GrMa2, MaBr], where the potentials are much more general, but the energy considered is lower than that of the crossing. Actually, in this last situation the width of the resonances can be estimated by a tunneling effect through a potential barrier, and it is exponentially small (in the parameter h). In our case, on the contrary, it is expected that the width of resonances is much larger and of polynomial order of h.

The key point will be the connection problem at the crossing point of the asymptotic solutions defined on the right and left of this point respectively.

However, the recently developed techniques, such as complex exact WKB method (see for example [Vo]) or microlocal method (see for example [HeSj, Ma]), are not easily applied to such a problem because of the degeneracy at the crossing point. More precisely, the determinant of the principal symbol of our operator is of type $\xi^4 - x^2$ (ξ denotes the dual variable to the independent variable x), and two bicharacteristic curves intersect at $(0,0)$ tangentially.

We make use of the global solutions constructed in [Ya] for the scalar equations to construct resolvents to our system. By a careful estimate of these operators, we can construct solutions to the system by iteration on the left and the right of the crossing point. Then the resonances will be the zeros of the wronskian of these solutions. The advantage of Yafaev's solutions consists in the knowledge of the asymptotic behavior at the crossing point.

In this note, we sketch only the process of the computations. For the sake of simplicity, we assume that the derivatives of the two potentials at the crossing point are 1 and -1 (see assumption (A3) of the next section). For the results in the general case and for the proof of the estimates used in this note, we refer the readers to [FMW1] and [FMW2].

§ 2. Assumptions and results

We consider a 2×2 Schrödinger operator of the type,

$$Pu = Eu, \qquad P = \begin{pmatrix} P_1 & hW \\ hW^* & P_2 \end{pmatrix}, \tag{2.1}$$

where $P_j = h^2 D_x^2 + V_j(x)$ $(j = 1, 2)$ with $D_x = -i\frac{d}{dx}$, $W = W(x, hD_x)$ is a first order semiclassical differential operator, and W^* is the formal adjoint of W.

We suppose the following conditions on the potentials $V_1(x), V_2(x)$ and on the interaction $W(x, hD_x)$:

(A1) $V_1(x)$, $V_2(x)$ are real-valued analytic functions on $\mathbb{R}$, and extend to holomorphic functions in the complex domain,

$$\Gamma = \{x \in \mathbb{C}\,;\, |\mathrm{Im}\, x| < \delta_0 \langle \mathrm{Re}\, x \rangle\},$$

where $\delta_0 > 0$ is a constant, and $\langle t \rangle := (1 + |t|^2)^{1/2}$.
(A2) For $j = 1, 2$, V_j admit limits as $\mathrm{Re}\, x \to \pm\infty$ in Γ, and they satisfy,

$$\lim_{\substack{\mathrm{Re}\, x \to -\infty \\ x \in \Gamma}} V_1(x) > 0\,; \lim_{\substack{\mathrm{Re}\, x \to -\infty \\ x \in \Gamma}} V_2(x) > 0\,;$$
$$\lim_{\substack{\mathrm{Re}\, x \to +\infty \\ x \in \Gamma}} V_1(x) > 0\,; \lim_{\substack{\mathrm{Re}\, x \to +\infty \\ x \in \Gamma}} V_2(x) < 0.$$

(A3) There exists a negative number $x^* < 0$ such that,

- $V_1 > 0$ and $V_2 > 0$ on $(-\infty, x^*)$;
- $V_1 < 0 < V_2$ on $(x^*, 0)$;
- $V_2 < 0 < V_1$ on $(0, +\infty)$,

and one has,

$$V_1'(x^*) < 0, \qquad V_1'(0) = 1, \qquad V_2'(0) = -1.$$

(A4) The interaction $W(x, hD_x)$ is a differential operator of the form,

$$W(x, hD_x) = r_0(x) + ir_1(x)hD_x,$$

where $r_0(x)$ and $r_1(x)$ are bounded analytic functions on Γ, and $r_0(x)$ is real on $\mathbb{R}$.

Notice that, in a neighborhood of $E = 0$, the scalar operator P_1 has eigenvalues, while P_2 has only essential spectrum. Hence, if the interaction W is absent, the matrix-valued operator P has, of course, embedded eigenvalues in the essential spectrum. But if W is present, it is expected that there exist, instead of embedded eigenvalues, resonances close to them in the lower half complex plane of the energy.

The resonances of P are defined, e.g., as the values $E \in \mathbb{C}$ such that the equation $Pu = Eu$ has a non-trivial outgoing solution u, that is, a non-identically vanishing solution such that, for some $\theta > 0$ sufficiently small, the function $x \mapsto u(xe^{i\theta})$ is in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ (see, e.g., [AgCo, ReSi]). Equivalently, the resonances are the eigenvalues of the operator P acting on $L^2(\mathbb{R}_\theta) \oplus L^2(\mathbb{R}_\theta)$, where $\mathbb{R}_\theta$ is a complex distortion of $\mathbb{R}$ that coincides with $e^{i\theta}\mathbb{R}$ for $x \gg 1$ (see, e.g., [HeMa]). We denote by $\mathrm{Res}(P)$ the set of these resonances.

For $E \in \mathbb{C}$ small enough, we define the action,

$$\mathcal{A}(E) := \int_{x_1^*(E)}^{x_1(E)} \sqrt{E - V_1(t)}\, dt,$$

where $x_1^*(E)$ (respectively $x_1(E)$) is the unique solution of $V_1(x) = E$ close to x^* (respectively close to 0), and it is well-known that, in this situation, $\mathcal{A}(E)$ is an analytic function of E near 0.

We also fix $C_0 > 0$ arbitrarily large, and we study the resonances of P lying in the set $\mathcal{D}_h(C_0)$ given by,

$$\mathcal{D}_h(C_0) := [-C_0h^{2/3}, C_0h^{2/3}] - i[0, C_0h]. \tag{2.2}$$

For $h > 0$ and $k \in \mathbb{Z}$, we set,

$$\lambda_k(h) := \frac{-\mathcal{A}(0) + (k + \frac{1}{2})\pi h}{\mathcal{A}'(0)h^{2/3}}. \tag{2.3}$$

Recall that the Bohr-Sommerfeld quantization condition of eigenvalues for the scalar operator P_1 reads

$$\mathcal{A}(E) = (k + \frac{1}{2})\pi h + \mathcal{O}(h^2).$$

Then the $\lambda_k(h)h^{2/3}$'s are approximate eigenvalues of P_1 near 0. We will find resonances close to these real values.

Theorem 2.1 ([FMW1]). *Assume* **(A1)**-**(A4)**. *For $h > 0$ small enough, one has,*

$$\mathrm{Res}\,(P) \cap \mathcal{D}_h(C_0) = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h(C_0),$$

where the $E_k(h)$'s are complex numbers that satisfy,

$$\mathrm{Re}\, E_k(h) = \lambda_k(h)h^{\frac{2}{3}} - \frac{\mathcal{A}''(0)}{2\mathcal{A}'(0)}\lambda_k(h)^2h^{\frac{4}{3}} + \mathcal{O}(h^{\frac{5}{3}}), \tag{2.4}$$

$$\mathrm{Im}\, E_k(h) = -\frac{\pi^2 r_0(0)^2}{2^{2/3}\mathcal{A}'(0)}\left(\mathrm{Ai}\,(-2^{\frac{2}{3}}\lambda_k(h))\right)^2 h^{\frac{5}{3}} + \mathcal{O}(h^2), \tag{2.5}$$

uniformly as $h \to +0$, where Ai *stands for the Airy function.*

Theorem 2.2 ([FMW2]). *Assume moreover that $r_0(x) = 0$ and $r_1(x)$ is real on $\mathbb{R}$. Then for $h > 0$ small enough, one has,*

$$\mathrm{Res}\,(P) \cap \mathcal{D}_h(C_0) = \{E_k(h); k \in \mathbb{Z}\} \cap \mathcal{D}_h(C_0),$$

where the $E_k(h)$'s are complex numbers that satisfy,

$$\mathrm{Re}\, E_k(h) = \lambda_k(h)h^{\frac{2}{3}} - \frac{\mathcal{A}''(0)}{2\mathcal{A}'(0)}\lambda_k(h)^2h^{\frac{4}{3}} - \frac{\mathcal{A}^{(3)}(0)}{6\mathcal{A}'(0)}\lambda_k(h)^3h^{\frac{6}{3}} + \mathcal{O}(h^{\frac{7}{3}}), \tag{2.6}$$

$$\mathrm{Im}\, E_k(h) = -\frac{\pi^2 r_1(0)^2}{2^{4/3}\mathcal{A}'(0)}\left(\mathrm{Ai}\,'(-2^{\frac{2}{3}}\lambda_k(h))\right)^2 h^{\frac{7}{3}} + \mathcal{O}(h^{\frac{8}{3}}), \tag{2.7}$$

uniformly as $h \to +0$.

§ 3. Outline of the proof

We fix a sufficiently small $\theta > 0$. Let

$$I_L := (-\infty, 0] = \mathbb{R}_- \ ; \ I_R^\theta := F_\theta([0, +\infty)) = F_\theta(\mathbb{R}_+) \ ; \ F_\theta(x) := x + i\theta f(x)$$

where $f \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$, $f(x) = x$ for x large enough, $f(x) = 0$ for $x \in [0, x_\infty]$ for some $x_\infty > 0$, and f is chosen in such a way that, for any $x \geq x_\infty$, one has,

$$\operatorname{Im} \int_{x_\infty}^{F_\theta(x)} \sqrt{E - V_2(t)} dt \geq -Ch, \tag{3.1}$$

with some positive constant C (see [FMW1]).

The linear space V of solutions to the system (2.1) is of dimension four. The outgoing solutions on $\mathbb{R}_+$, i.e. solutions in $L^2(I_R^\theta) \oplus L^2(I_R^\theta)$ form a two dimensional subspace $V_R = V \cap (L^2(I_R^\theta) \oplus L^2(I_R^\theta))$, and the outgoing solutions on I_L also constitute a two dimensional subspace V_L.

Then E is a resonance if and only if the intersection $V_R \cap V_L$ is at least one dimensional. In other words, the quantization condition of resonances can be written in the form

$$\mathcal{W}_0(E) := \mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}) = 0, \tag{3.2}$$

where the couple $(w_{1,L}, w_{2,L})$ (resp. $(w_{1,R}, w_{2,R})$) is a basis of V_L (resp. V_R) and $\mathcal{W}(w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R})$ is the wronskian, i.e. the determinant of the 4×4 matrix

$$\begin{pmatrix} w_{1,L} & w_{2,L} & w_{1,R} & w_{2,R} \\ \partial_x w_{1,L} & \partial_x w_{2,L} & \partial_x w_{1,R} & \partial_x w_{2,R} \end{pmatrix}.$$

We will construct such solutions $w_{1,L}, w_{2,L}, w_{1,R}, w_{2,R}$ and compute the asymptotic formula of their wronskian. This formula will give the precise estimate of the location of resonances given in the above theorems.

§ 3.1. Solutions to the scalar equations

Before constructing bases of solutions to the system, we need to study the fundamental solutions to the scalar operators $P_j - E$, $j = 1, 2$.

First we construct solutions to the homogeneous equation $(P_j - E)u = 0$. Employing a classical method as in [Ya], we obtain four solutions $u_{1,R}^\pm, u_{1,L}^\pm$ to $(P_1 - E)u = 0$ satisfying the asymptotic behavior

$$u_{1,R}^\pm(x) \sim (1 + \mathcal{O}(h)) \frac{h^{\frac{1}{6}}}{\sqrt{\pi}} (V_1(x) - E)^{-\frac{1}{4}} e^{\pm \int_{x_1(E)}^x \sqrt{V_1(t) - E}\, dt/h}, \quad (x \to +\infty),$$

$$u_{1,L}^{\pm}(x) \sim (1+\mathcal{O}(h))\frac{h^{\frac{1}{6}}}{\sqrt{\pi}}(V_1(x)-E)^{-\frac{1}{4}}e^{\mp\int_{x_1^*(E)}^{x}\sqrt{V_1(t)-E}\,dt/h}, \quad (x\to-\infty),$$

and four solutions $u_{2,R}^{\pm}, u_{2,L}^{\pm}$ to $(P_2-E)u=0$ satisfying the asymptotic behavior

$$u_{2,R}^{\pm}(x) \sim (1+\mathcal{O}(h))\frac{h^{\frac{1}{6}}}{\sqrt{\pi}}(E-V_2(x))^{-\frac{1}{4}}e^{\pm i\int_{x_2(E)}^{x}\sqrt{E-V_2(t)}\,dt/h}, \quad (x\to+\infty),$$

$$u_{2,L}^{\pm}(x) \sim (1+\mathcal{O}(h))\frac{h^{\frac{1}{6}}}{\sqrt{\pi}}(V_2(x)-E)^{-\frac{1}{4}}e^{\mp\int_{x_2(E)}^{x}\sqrt{V_2(t)-E}\,dt/h}, \quad (x\to-\infty),$$

where $x_j(E)$ is the zero of $V_j(x)-E$ near 0 and $x_1^*(E)$ is the zero of $V_1(x)-E$ near x^*.

Yafaev's method consists in the reduction to the Airy equation near the turning points: for example near $x_1(E)$, we define for E real and a point x_0 inside the well $(x^*, 0)$,

$$\begin{aligned}\xi_1(x;E) &= \left(\frac{3}{2}\int_{x_1(E)}^{x}\sqrt{V_1(t)-E}\,dt\right)^{2/3} && \text{when } x\ge x_1(E);\\ \xi_1(x;E) &= -\left(\frac{3}{2}\int_{x}^{x_1(E)}\sqrt{E-V_1(t)}\,dt\right)^{2/3} && \text{when } x_0\le x\le x_1(E).\end{aligned} \tag{3.3}$$

Thanks to the analyticity, this change of variable extends to complex x and E. Setting $t:=h^{-2/3}\xi_1(x)$ and $f(t):=\xi_1'(x)^{1/2}u(x)$, the equation $(P_1-E)u=0$ becomes,

$$-f''(t)+tf(t)=R(t)f(t), \tag{3.4}$$

with,

$$\begin{aligned}&R(t)=h^{\frac{4}{3}}p(h^{\frac{2}{3}}t);\\ &p(x):=\left[(\xi_1'(x))^{-\frac{1}{2}}\right]''(\xi_1'(x))^{-\frac{3}{2}}=\mathcal{O}(1+|\xi_1(x)|)^{-2}.\end{aligned} \tag{3.5}$$

The advantage of this method is to know well the asymptotic behavior as $h\to+0$ of the solutions near the turning points. For example, for $u_{1,R}^{\pm}$, we have, as $h\to+0$,

$$\begin{aligned}u_{1,R}^{-}(x) &= 2(\xi_1'(x))^{-\frac{1}{2}}\operatorname{Ai}(h^{-\frac{2}{3}}\xi_1(x))(1+\mathcal{O}(h))\\ &\qquad\qquad \text{on } [x_0,+\infty)\cap\{\operatorname{Re}\xi_1(x)\ge 0\};\\ u_{1,R}^{-}(x) &= 2(\xi_1'(x))^{-\frac{1}{2}}\operatorname{Ai}(h^{-\frac{2}{3}}\xi_1(x))+\mathcal{O}(h(1+h^{-2/3}|\xi_1(x)|)^{-\frac{1}{4}})\\ &\qquad\qquad \text{on } [x_0,+\infty)\cap\{\operatorname{Re}\xi_1(x)\le 0\};\\ u_{1,R}^{+}(x) &= (\xi_1'(x))^{-\frac{1}{2}}\operatorname{Bi}(h^{-\frac{2}{3}}\xi_1(x))(1+\mathcal{O}(h))\\ &\qquad\qquad \text{on } [x_0,+\infty)\cap\{\operatorname{Re}\xi_1(x)\ge 0\};\\ u_{1,R}^{+}(x) &= (\xi_1'(x))^{-\frac{1}{2}}\operatorname{Bi}(h^{-\frac{2}{3}}\xi_1(x))+\mathcal{O}(h(1+h^{-2/3}|\xi_1(x)|)^{-\frac{1}{4}})\\ &\qquad\qquad \text{on } [x_0,+\infty)\cap\{\operatorname{Re}\xi_1(x)\le 0\}.\end{aligned}$$

We have similar formulae for other solutions. For $u^{\pm}_{2,L}$, we have

$$u^{+}_{2,L} \sim (\xi_2')^{-\frac{1}{2}}\mathrm{Bi}\,(-h^{-\frac{2}{3}}\xi_2); \qquad u^{-}_{2,L} \sim 2(\xi_2')^{-\frac{1}{2}}\mathrm{Ai}\,(-h^{-\frac{2}{3}}\xi_2).$$

The solutions $u^{\pm}_{2,R}$ can be expressed in terms of $u^{\pm}_{2,L}$

$$u^{+}_{2,R} \sim \sqrt{2}e^{i\pi/4}(\frac{1}{2}u^{-}_{2,L} + iu^{+}_{2,L}); \qquad u^{-}_{2,R} \sim \frac{1}{\sqrt{2}}e^{i\pi/4}(\frac{1}{2}u^{-}_{2,L} - iu^{+}_{2,L}),$$

and also $u^{\pm}_{1,L}$ can be expressed in terms of $u^{\pm}_{1,R}$

$$u^{+}_{1,L} \sim \frac{1}{2}(\cos\frac{\mathcal{A}}{h})u^{-}_{1,R} - (\sin\frac{\mathcal{A}}{h})u^{+}_{1,R}; \qquad u^{-}_{1,L} \sim (\sin\frac{\mathcal{A}}{h})u^{-}_{1,R} + 2(\cos\frac{\mathcal{A}}{h})u^{+}_{1,R}.$$

Remark that the action $\mathcal{A}$ appears here (notice that the phase of $u^{\pm}_{1,L}$ has its base point at $x_1^*(E)$).

It is also easy to compute the wronskians between these solutions: Let $\widetilde{\mathcal{W}}(f,g)$ stand for $h^{2/3}\det(f,g)$ for two column vectors f,g of degree 2.

$$\widetilde{\mathcal{W}}(u^{-}_{j,L}, u^{+}_{j,L}) \sim \frac{-2}{\pi} \quad ; \quad \widetilde{\mathcal{W}}(u^{-}_{j,R}, u^{+}_{j,R}) \sim \frac{2}{\pi};$$

$$\widetilde{\mathcal{W}}(u^{-}_{1,L}, u^{-}_{1,R}) \sim \frac{-4}{\pi}\cos\frac{\mathcal{A}}{h} \quad ; \quad \widetilde{\mathcal{W}}(u^{-}_{2,L}, u^{-}_{2,R}) \sim \frac{i\sqrt{2}}{\pi}e^{i\frac{\pi}{4}};$$

$$\widetilde{\mathcal{W}}(u^{\pm}_{1,L}, u^{\mp}_{1,R}) = \frac{2}{\pi}\sin\frac{\mathcal{A}}{h};$$

$$\widetilde{\mathcal{W}}(u^{-}_{2,L}, u^{+}_{2,R}) \sim \frac{-2i\sqrt{2}}{\pi}e^{i\frac{\pi}{4}} \quad ; \quad \widetilde{\mathcal{W}}(u^{+}_{2,L}, u^{-}_{2,R}) \sim \frac{1}{\pi\sqrt{2}}e^{i\frac{\pi}{4}}.$$

§ 3.2. Resolvents to the scalar equation

Using the solutions defined in the previous section, we construct fundamental solutions $K_{j,L}$, $j=1,2$ on I_L and $K_{j,R}$, $j=1,2$ on I_R^θ:

$$\begin{aligned} K_{j,L}[v](x) := &\frac{u^{+}_{j,L}(x)}{h^2\mathcal{W}[u^{+}_{j,L}, u^{-}_{j,L}]}\int_{-\infty}^{x} u^{-}_{j,L}(t)v(t)\,dt \\ &+ \frac{u^{-}_{j,L}(x)}{h^2\mathcal{W}[u^{+}_{j,L}, u^{-}_{j,L}]}\int_{x}^{0} u^{+}_{j,L}(t)v(t)\,dt; \end{aligned} \tag{3.6}$$

$$\begin{aligned} K_{j,R}[v](x) := &\frac{u^{-}_{j,R}(x)}{h^2\mathcal{W}[u^{-}_{j,R}, u^{+}_{j,R}]}\int_{0}^{x} u^{+}_{j,R}(t)v(t)\,dt \\ &+ \frac{u^{+}_{j,R}(x)}{h^2\mathcal{W}[u^{-}_{j,R}, u^{+}_{j,R}]}\int_{x}^{+\infty} u^{-}_{j,R}(t)v(t)\,dt, \end{aligned} \tag{3.7}$$

where the integral runs over I_L and I_R^θ respectively.

Let $C_b^0(I_L)$ and $C_b^0(I_R^\theta)$ be the space of bounded continuous functions on I_L and I_R^θ respectively. The above operators act on these function spaces, and satisfy $(P_j - E)K_{j,L} = Id$ and $(P_j - E)K_{j,R} = Id$ respectively. Moreover one can prove the following estimates:

Proposition 3.1. *As h goes to 0, one has uniformly,*

$$\| hK_{2,L}W^* \|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h^{\frac{1}{3}}); \qquad \| h^2K_{1,L}WK_{2,L}W^* \|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h^{\frac{2}{3}});$$
$$\| hK_{1,R}W \|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h^{\frac{1}{3}}); \qquad \| h^2K_{2,R}W^*K_{1,R}W \|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h^{\frac{2}{3}}).$$

If moreover $r_0(x) \equiv 0$, one has

$$\| hK_{2,L}W^* \|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h^{\frac{2}{3}}); \qquad \| h^2K_{1,L}WK_{2,L}W^* \|_{\mathcal{L}(C_b^0(I_L))} = \mathcal{O}(h);$$
$$\| hK_{1,R}W \|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h^{\frac{2}{3}}); \qquad \| h^2K_{2,R}W^*K_{1,R}W \|_{\mathcal{L}(C_b^0(I_R^\theta))} = \mathcal{O}(h).$$

§ 3.3. Solutions to the system

Set $M_L := h^2K_{1,L}WK_{2,L}W^*$ and $M_R := h^2K_{2,R}W^*K_{1,R}W$. Thanks to Proposition 3.1, we can define the following four vector-valued functions as Neumann series for small enough h;

$$w_{1,L} := \begin{pmatrix} \sum_{j\geq 0} M_L^j u_{1,L}^- \\ -hK_{2,L}W^* \sum_{j\geq 0} M_L^j u_{1,L}^- \end{pmatrix}, \tag{3.8}$$

$$w_{2,L} := \begin{pmatrix} -\sum_{j\geq 0} M_L^j (hK_{1,L}Wu_{2,L}^-) \\ u_{2,L}^- + hK_{2,L}W^* \sum_{j\geq 0} M_L^j (hK_{1,L}Wu_{2,L}^-) \end{pmatrix}, \tag{3.9}$$

$$w_{1,R} := \begin{pmatrix} u_{1,R}^- + hK_{1,R}W \sum_{j\geq 0} M_R^j (hK_{2,R}W^*u_{1,R}^-) \\ -\sum_{j\geq 0} M_R^j (hK_{2,R}W^*u_{1,R}^-) \end{pmatrix}, \tag{3.10}$$

$$w_{2,R} := \begin{pmatrix} -hK_{1,R}W \sum_{j\geq 0} M_R^j u_{2,R}^- \\ \sum_{j\geq 0} M_R^j u_{2,R}^- \end{pmatrix}. \tag{3.11}$$

It is easy to see that they are solutions to the system (2.1) and that

$$w_{j,L} \in L^2(I_L) \oplus L^2(I_L) \quad ; \quad w_{j,R} \in L^2(I_R^\theta) \oplus L^2(I_R^\theta).$$

In order to get the leading term of the imaginary part of resonances, it will be necessary to compute the asymptotics of these solutions up to errors of $\mathcal{O}(h)$ in the case of elliptic interaction and of $\mathcal{O}(h^{5/3})$ in the case of vector field interaction (i.e.

$r_0(x) \equiv 0$). This means to compute, for example for $w_{1,L}$, two terms $u_{1,L}^- + M_L u_{1,L}^-$ for the first element, and one term $-hK_{2,L}W^* u_{1,L}^-$ for the second element.

Substituting $x = 0$ to these solutions or their derivatives, we obtain the following asymptotic formulae.

Proposition 3.2. *For $j = 1, 2$ and $S = L, R$, we have, uniformly as $h \to +0$,*

$$\begin{aligned} w_{1,S}(0) &= \begin{bmatrix} u_{1,S}^-(0) + \beta_{1,S} u_{1,S}^+(0) \\ \alpha_{1,S} u_{2,S}^+(0) \end{bmatrix} + \mathcal{O}(h); \\ \widetilde{\partial} w_{1,S}(0) &= \begin{bmatrix} \widetilde{\partial} u_{1,S}^-(0) + \beta_{1,S} \widetilde{\partial} u_{1,S}^+(0) \\ \alpha_{1,S} \widetilde{\partial} u_{2,S}^+(0) \end{bmatrix} + \mathcal{O}(h), \end{aligned} \tag{3.12}$$

$$\begin{aligned} w_{2,S}(0) &= \begin{bmatrix} \alpha_{2,S} u_{1,S}^+(0) \\ u_{2,S}^-(0) + \beta_{2,S} u_{2,S}^+(0) \end{bmatrix} + \mathcal{O}(h); \\ \widetilde{\partial} w_{2,S}(0) &= \begin{bmatrix} \alpha_{2,S} \widetilde{\partial} u_{1,S}^+(0) \\ \widetilde{\partial} u_{2,S}^-(0) + \beta_{2,S} \widetilde{\partial} u_{2,S}^+(0) \end{bmatrix} + \mathcal{O}(h). \end{aligned} \tag{3.13}$$

Here, $\widetilde{\partial}$ stands for $h^{2/3}\partial$ and $\alpha_{j,S}$ and $\beta_{j,S}$ are complex numbers defined by

$$\alpha_{1,L} = \frac{-\int_{-\infty}^0 u_{2,L}^-(t)(W^* u_{1,L}^-)(t)dt}{h\mathcal{W}(u_{2,L}^+, u_{2,L}^-)} \quad ; \quad \beta_{1,L} = \frac{\int_{-\infty}^0 u_{1,L}^-(t)(W r_{1,L})(t)dt}{h\mathcal{W}(u_{1,L}^+, u_{1,L}^-)};$$

$$\alpha_{2,L} = \frac{-\int_{-\infty}^0 u_{1,L}^-(t)(W u_{2,L}^-)(t)dt}{h\mathcal{W}(u_{1,L}^+, u_{1,L}^-)} \quad ; \quad \beta_{2,L} = \frac{\int_{-\infty}^0 u_{2,L}^-(t)(W^* r_{2,L})(t)dt}{h\mathcal{W}(u_{2,L}^+, u_{2,L}^-)};$$

$$\alpha_{1,R} = \frac{-\int_0^{+\infty} u_{2,R}^-(t)(W^* u_{1,R}^-)(t)dt}{h\mathcal{W}(u_{2,R}^-, u_{2,R}^+)} \quad ; \quad \beta_{1,R} = \frac{\int_0^{+\infty} u_{1,R}^-(t)(W r_{1,R})(t)dt}{h\mathcal{W}(u_{1,R}^-, u_{1,R}^+)};$$

$$\alpha_{2,R} = \frac{-\int_0^{+\infty} u_{1,R}^-(t)(W u_{2,R}^-)(t)dt}{h\mathcal{W}(u_{1,R}^-, u_{1,R}^+)} \quad ; \quad \beta_{2,R} = \frac{\int_0^{+\infty} u_{2,R}^-(t)(W^* r_{2,R})(t)dt}{h\mathcal{W}(u_{2,R}^-, u_{2,R}^+)},$$

where we have set, for $S = L, R$,

$$r_{1,S} := hK_{2,S}W^* u_{1,S}^-, \qquad r_{2,S} := hK_{1,S}W u_{2,S}^-,$$

and where, in the case $S = R$, the integrals run over I_R^θ.

If moreover $r_0(x) \equiv 0$, then the remainder terms in (3.12) and (3.13) are $\mathcal{O}(h^{5/3})$.

For the constants appearing in the previous proposition, we have the following estimates, which are necessary and sufficient to get Theorems 2.1 and 2.2. Let $E = \rho h^{2/3} \in \mathcal{D}_h(C_0)$. We see that they are expressed in terms of functions of $t = \mathrm{Re}\rho$

defined as integrals of Airy functions:

$$\begin{aligned}
\mu_A(t) &:= \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Ai}\,(-y-t)dy, & \mu_B(t) &:= \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Bi}\,(-y-t)dy,\\
\nu_A(t) &:= \int_0^\infty \mathrm{Ai}\,'(y-t)\mathrm{Ai}\,(-y-t)dy, & \nu_A^\dagger(t) &:= \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Ai}\,'(-y-t)dy,\\
\nu_B(t) &:= \int_0^\infty \mathrm{Ai}\,'(y-t)\mathrm{Bi}\,(-y-t)dy, & \nu_B^\dagger(t) &:= \int_0^\infty \mathrm{Ai}\,(y-t)\mathrm{Bi}\,'(-y-t)dy.
\end{aligned}$$

These integrals are all well defined thanks to the exponential decay of $\mathrm{Ai}\,(y-t)$ and $\mathrm{Ai}\,'(y-t)$ and the polynomial growth of $\mathrm{Ai}\,(-y-t)$, $\mathrm{Ai}\,'(-y-t)$, $\mathrm{Bi}\,(-y-t)$, $\mathrm{Bi}\,('-y-t)$ as $y\to+\infty$.

Among these quantities, those with suffix A will play an important role in the asymptotics of the imaginary part of resonances. As is easily seen, they are related with each other by identities

$$\nu_A(t)+\nu_A^\dagger(t) = -\mu_A'(t), \quad \nu_A(t)-\nu_A^\dagger(t) = -\mathrm{Ai}\,(-t)^2.$$

Furthermore, $\mu_A(t)$ can in fact be written explicitly by

$$\mu_A(t) = 2^{-4/3}\mathrm{Ai}\,(-2^{2/3}t).$$

Hence we also have

$$\begin{aligned}
\nu_A(t) &= 2^{-5/3}\mathrm{Ai}\,'(-2^{2/3}t) - 2^{-1}\mathrm{Ai}\,(-t)^2,\\
\nu_A^\dagger(t) &= 2^{-5/3}\mathrm{Ai}\,'(-2^{2/3}t) + 2^{-1}\mathrm{Ai}\,(-t)^2.
\end{aligned}$$

Proposition 3.3. *Let $E=\rho h^{2/3}\in\mathcal{D}_h(C_0)$. As $h\to+0$, one has,*

$$\begin{aligned}
&\alpha_{j,L} = -2h^{1/3}\pi r_0(0)\left(\mu_A(\mathrm{Re}\,\rho)\sin\frac{\mathcal{A}(E)}{h} + \mu_B(\mathrm{Re}\,\rho)\cos\frac{\mathcal{A}(E)}{h}\right) + \mathcal{O}(h^{2/3}),\\
&\alpha_{j,R} = -\frac{h^{1/3}\pi r_0(0)e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(\mu_A(\mathrm{Re}\,\rho) - i\mu_B(\mathrm{Re}\,\rho)\right) + \mathcal{O}(h^{2/3}), \quad (j=1,2),\\
&\mathrm{Im}\,\beta_{1,R} = \pi^2 r_0(0)^2 h^{2/3}\left(\mu_A(\mathrm{Re}\,\rho)^2 + \mu_B(\mathrm{Re}\,\rho)^2\right) + \mathcal{O}(h),\\
&\mathrm{Im}\,\beta_{1,L} = \mathcal{O}(h), \quad \beta_{j,S} = \mathcal{O}(h^{2/3}), \quad (j=1,2,\ S=L,R).
\end{aligned}$$

If moreover $r_0(x) \equiv 0$ and $r_1(x)$ is real-valued on $\mathbb{R}$, one has, as $h \to +0$,

$$
\begin{aligned}
\alpha_{1,R} &= \frac{e^{\pi i/4}}{\sqrt{2}} \pi r_1(0) h^{2/3} \Big(\nu_A(\operatorname{Re}\rho) - i\nu_B(\operatorname{Re}\rho) \Big) + \mathcal{O}(h), \\
\alpha_{2,R} &= \frac{e^{\pi i/4}}{\sqrt{2}} \pi r_1(0) h^{2/3} \Big(\nu_A^\dagger(\operatorname{Re}\rho) - i\nu_B^\dagger(\operatorname{Re}\rho) \Big) + \mathcal{O}(h), \\
\alpha_{1,L} &= 2\pi r_1(0) h^{2/3} \left\{ \left(\sin \frac{\mathcal{A}}{h} \right) \nu_A^\dagger(\operatorname{Re}\rho) + \left(\cos \frac{\mathcal{A}}{h} \right) \nu_B^\dagger(\operatorname{Re}\rho) \right\} + \mathcal{O}(h), \\
\alpha_{2,L} &= 2\pi r_1(0) h^{2/3} \left\{ \left(\sin \frac{\mathcal{A}}{h} \right) \nu_A(\operatorname{Re}\rho) + \left(\cos \frac{\mathcal{A}}{h} \right) \nu_B(\operatorname{Re}\rho) \right\} + \mathcal{O}(h), \\
\operatorname{Im}\beta_{1,R} &= \pi^2 r_1(0)^2 h^{4/3} \Big(\nu_A(\operatorname{Re}\rho)\nu_A^\dagger(\operatorname{Re}\rho) + \nu_B(\operatorname{Re}\rho)\nu_B^\dagger(\operatorname{Re}\rho) \Big) + \mathcal{O}(h^{5/3}), \\
\operatorname{Im}\beta_{1,L} &= \mathcal{O}(h^{5/3}), \quad \beta_{j,S} = \mathcal{O}(h^{4/3}), \quad (j = 1, 2,\ S = L, R).
\end{aligned}
$$

§ 3.4. Wronskian and quantization condition

Now we are ready to compute the wronskian $\mathcal{W}_0(E, h)$. Since it is independent of x, we compute it at $x = 0$. In terms of the constants defined in Proposition 3.2 and wronskians of solutions to the scalar equations, we can write the wronskian $\mathcal{W}_0(E, h)$ as follows.

$$
\begin{aligned}
\mathcal{W}_0(E,h) =& \widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^-)\widetilde{\mathcal{W}}(u_{2,L}^-, u_{2,R}^-) \\
&+ \beta_{2,R}\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^-)\widetilde{\mathcal{W}}(u_{2,L}^-, u_{2,R}^+) \\
&+ \beta_{2,L}\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^-)\widetilde{\mathcal{W}}(u_{2,L}^+, u_{2,R}^-) \\
&+ \alpha_{1,R}\alpha_{2,R}\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^+)\widetilde{\mathcal{W}}(u_{2,R}^+, u_{2,L}^-) \\
&+ \alpha_{1,R}\alpha_{2,L}\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,L}^+)\widetilde{\mathcal{W}}(u_{2,R}^-, u_{2,R}^+) \\
&+ \beta_{1,R}\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^+)\widetilde{\mathcal{W}}(u_{2,L}^-, u_{2,R}^-) \\
&+ \alpha_{1,L}\alpha_{2,R}\widetilde{\mathcal{W}}(u_{2,L}^+, u_{2,L}^-)\widetilde{\mathcal{W}}(u_{1,R}^+, u_{1,R}^-) \\
&+ \alpha_{1,L}\alpha_{2,L}\widetilde{\mathcal{W}}(u_{2,L}^+, u_{2,R}^-)\widetilde{\mathcal{W}}(u_{1,R}^-, u_{1,L}^+) \\
&+ \beta_{1,L}\widetilde{\mathcal{W}}(u_{1,L}^+, u_{1,R}^-)\widetilde{\mathcal{W}}(u_{2,L}^-, u_{2,R}^-) + \mathcal{O}(h).
\end{aligned}
$$

In particular, if $r_0(x) \equiv 0$, the remainder term is of $\mathcal{O}(h^{5/3})$.

This wronskian formula together with (3.2), Proposition 3.3 and the wronskian formulae in Section 2.1 lead to the quantization condition of resonances.

Notice here that the first term on the right hand side gives the principal term. It consists of two wronskians:

$$
\widetilde{\mathcal{W}}(u_{1,L}^-, u_{1,R}^-) \sim \frac{-4}{\pi} \cos \frac{\mathcal{A}(E)}{h} \quad ; \quad \widetilde{\mathcal{W}}(u_{2,L}^-, u_{2,R}^-) \sim \frac{i\sqrt{2}}{\pi} e^{i\frac{\pi}{4}}.
$$

The first one concerns the scalar operator P_1 and the condition that this wronskian is zero is the well-known Bohr-Sommerfeld quantization rule of eigenvalues of P_1. The same way, the second one gives the quantization condition of resonances of P_2 but it never vanishes reflecting the fact that P_2 is non-trapping.

All other terms are of $\mathcal{O}(h^{2/3})$ (resp. $\mathcal{O}(h^{4/3})$ in the case $r_0(x) = 0$). The imaginary part of these errors will give the precise asymptotics of the imaginary part of resonances.

Proposition 3.4. *$E = \rho h^{2/3} \in \mathcal{D}_h(C_0)$ is a resonance of P if and only if,*

$$\cos \frac{\mathcal{A}(E)}{h} = h^{2/3} \left(\sin \frac{\mathcal{A}(E)}{h} \right) F(E,h), \tag{3.14}$$

with

$$\begin{aligned} &\mathrm{Re}\, F(E,h) = \mathcal{O}(1), \\ &\mathrm{Im}\, F(E,h) = 4\pi^2 r_0(0)^2 \mu_A (\mathrm{Re}\rho)^2 + \mathcal{O}(h^{1/3}). \end{aligned}$$

If moreover $r_0(x) \equiv 0$ and $r_1(x)$ is real-valued, then $F(E,h) = h^{2/3} G(E,h)$ with

$$\begin{aligned} &\mathrm{Re}\, G(E,h) = \mathcal{O}(1), \\ &\mathrm{Im}\, G(E,h) = \pi^2 r_1(0)^2 \mu_A' (\mathrm{Re}\rho)^2 + \mathcal{O}(h^{1/3}). \end{aligned}$$

Theorems 2.1 and 2.2 are direct consequences of this proposition.

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